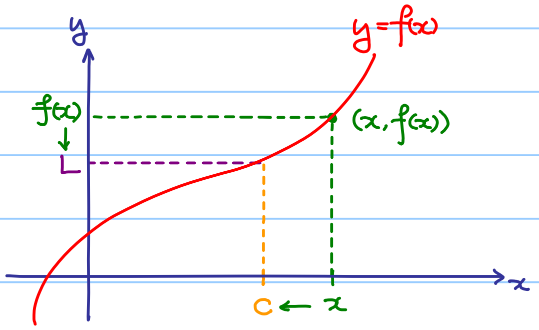
§3 Limits of Functions

3.1 Definition

Definition 3.1.1 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer† to c from both sides, then L is called the limit of $f(x)$ at c, and we write $\lim_{x \to c} f(x) = L$.



† Note: a little bit misleading!

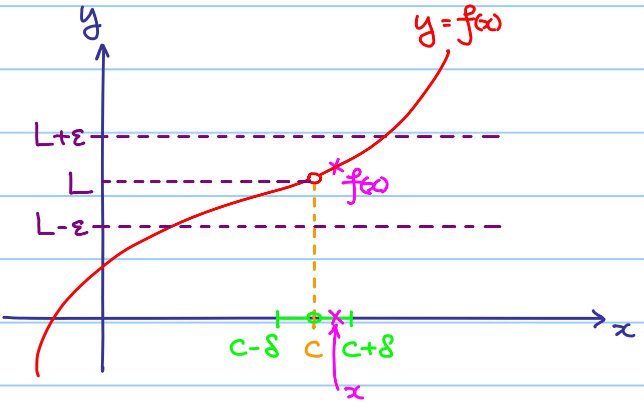
$f(c)$ may NOT equal to L, even it may be undefined!

Definition 3.1.2

Let $A \subseteq \mathbb{R}$, c be a cluster point of A and $f : A \to \mathbb{R}$ be a function.

$L \in \mathbb{R}$ is said to be the limit of f at the point c if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ $\forall x \in A$ with $0 < |x - c| < \delta$



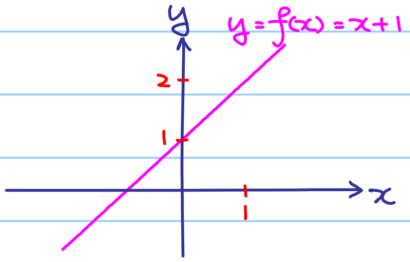
Meaning: No matter how small ε you give me,

I can always find $\delta > 0$ s.t. if x is a point with $\overset{\text{i.e. } x \neq c}{0 <} \text{dist}(x, c) < \delta$

then $f(x)$ lies in the ε-tunnel (ε-neighborhood of L)

Example 3.1.1

If $f(x) = x+1$, find $\lim_{x\to 1} f(x)$.



+

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x\to 1} f(x) = 2$.

Remarks:

1) + The table only gives an intuitive idea, but NOT a rigorous proof!

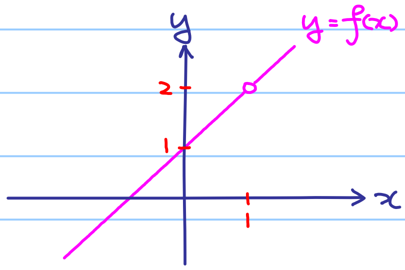
2) Do NOT regard as putting $x=1$ into $f(x)$ and get $f(1)=2$!

Example 3.1.2

Let $f(x)$ be a function defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

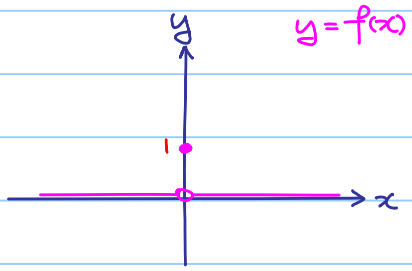
(But, we do NOT care what happens when $x=1$!)

We write $\lim_{x\to 1} f(x) = 2$

Compare with the previous example!

Example 3.1.3

Let $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



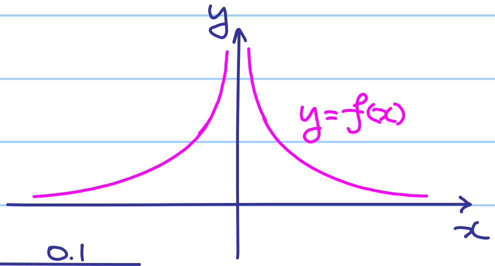
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0	0	0	1	0	0	0

Do NOT care !

$\lim_{x \to 0} f(x) = 0$ which does NOT equal to $f(0) = 1$.

Example 3.1.4

Let $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$.



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0.

$\therefore$ $\lim_{x \to 0} f(x)$ does NOT exist.

(But some still write $\lim_{x \to 0} f(x) = +\infty$.)

Theorem 3.1.1

1) If k is a constant, then $\lim_{x \to c} k = k$ regarded as constant function $f(x) = k$.

2) $\lim_{x \to c} x = c$.

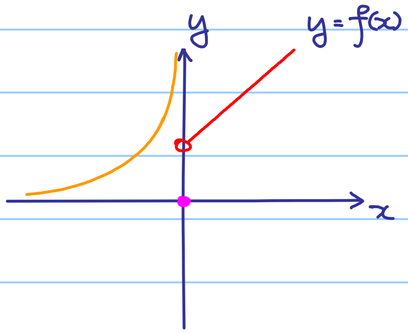
Definition 3.1.3 (Informal)

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from the right (resp. left) hand side, then L is called the right (resp. left) hand limit of $f(x)$ at c.

We denote it by $\lim_{x \to c^+} f(x) = L$ (resp. $\lim_{x \to c^-} f(x) = L$).

Example 3.1.5

$$\text{If } f(x) = \begin{cases} x+1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \frac{1}{x^2} & \text{if } x<0 \end{cases}$$



$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x+1 = 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x^2} \qquad (\text{does NOT exist})$$

$$f(0) = 0$$

Remark:

Right hand limit and left hand limit of a function at a point are NOT necessary to be the same!

Theorem 3.1.2

$$\lim_{x\to c} f(x) = L \iff \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

3.2 Algebraic Properties of Limits

Theorem 3.2.1

If both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist (Very important assumption!), then

(1) $\lim_{x\to c} f(x)+g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(2) $\lim_{x\to c} f(x)-g(x) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(3) $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

(4) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$.

Example 3.2.1

Find $\lim_{x\to2} 3x^2-5$.

Logically:

① $\lim_{x\to2} x = 2$, so $\lim_{x\to2} x^2 = \lim_{x\to2}(x\cdot x) \overset{\text{By (3)}}{=} \lim_{x\to2} x \cdot \lim_{x\to2} x = 2\cdot 2 = 4$

② $\lim_{x\to2} 3 = 3$, $\lim_{x\to2} x^2 = 4$, so $\lim_{x\to2} 3x^2 \overset{\text{By (3)}}{=} \lim_{x\to2} 3 \cdot \lim_{x\to2} x^2 = 3\cdot 4 = 12$

③ $\lim_{x\to2} 3x^2 = 12$, $\lim_{x\to2} 5 = 5$, so $\lim_{x\to2} 3x^2-5 \overset{\text{By (2)}}{=} \lim_{x\to2} 3x^2 - \lim_{x\to2} 5 = 12-5 = 7$

But what we write:

$$\begin{aligned}\lim_{x\to2} 3x^2-5 &= 3(\lim_{x\to2} x)^2 - 5\\ &= 3\cdot 2^2 - 5\\ &= 7\end{aligned}$$

Example 3.2.2

Find $\lim_{x\to1} \frac{3x^2-8}{x-2}$

$$\lim_{x\to1} \frac{3x^2-8}{x-2} = \frac{\lim_{x\to1} 3x^2-8}{\lim_{x\to1} x-2} = \frac{3(\lim_{x\to1} x)^2-8}{(\lim_{x\to1} x)-2} = \frac{3\cdot1^2-8}{1-2} = 5$$

Caution !

It seems that it makes no difference by putting $x=1$, and then

$$\lim_{x\to1} \frac{3x^2-8}{x-2} = \frac{3\cdot1^2-8}{1-2} = 5$$

But, think carefully! Let $f(x) = \frac{3x^2-8}{x-2}$, how do you know $\lim_{x\to1} f(x) = f(1)$?

Things will become clear when we discuss continuity of functions!

Example 3.2.3

Find $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2-3x+2}$

Note : $\lim\limits_{x\to 1} x^2-3x+2=0$, so we cannot use (4).

By (4)

$$\lim_{x\to 1}\frac{x^2-1}{x^2-3x+2}=\lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)(x-2)}\overset{\uparrow}{=}\lim_{x\to 1}\frac{x+1}{x-2}\overset{\downarrow}{=}\frac{\lim\limits_{x\to 1} x+1}{\lim\limits_{x\to 1} x-2}=\frac{2}{-1}=-2$$

$\because x\neq 1$

$\therefore x-1\neq 0$ and division can be done!

Example 3.2.4

Let $f:\mathbb{R}\setminus\{1\}\longrightarrow\mathbb{R}$ defined by $f(x)=\dfrac{\sqrt{x}-1}{x-1}$.

Find $\lim\limits_{x\to 1} f(x)$.

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

(Something like rationalization)

$$=\lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)}$$

$$=\lim_{x\to 1}\frac{1}{\sqrt{x}+1}$$

$$=\frac{1}{2}$$

Example 3.2.5

$$\lim_{x\to 0}\frac{1}{x}=\lim_{x\to 0}x\cdot\frac{1}{x^2}\overset{(*)}{=}\lim_{x\to 0}x\cdot\lim_{x\to 0}\frac{1}{x^2}=0\cdot\lim_{x\to 0}\frac{1}{x^2}=0$$

Anything wrong?

$\lim\limits_{x\to 0}\dfrac{1}{x^2}$ does NOT exist, so we cannot use (3) at (*).

3.3 Relation Between Limits of Sequences and Functions

Theorem 3.3.1

$\lim_{x \to c} f(x) = L \Leftrightarrow \forall$ sequence $\{a_n\}$ with

$a_n \neq c \ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = L$.

In fact, if we want to show $\lim_{x\to c} f(x) = L$, it is quite impossible to check infinitely many sequences. This statement is useful in reverse direction:

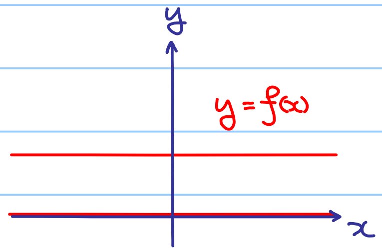
1) If $\exists \{a_n\}$ s.t. $\lim_{n\to\infty} a_n = c$, but $\lim_{n\to\infty} f(a_n)$ does NOT exist,
then $\lim_{x\to c} f(x)$ does NOT exist.

2) If $\exists \{a_n\}, \{b_n\}$ s.t. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$, but $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$
then $\lim_{x\to c} f(x)$ does NOT exist.

Example 3.3.1

Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



It seems the graph consists of two straight lines, but in fact infinitely many holes are there.

Consider sequences $\{a_n\}, \{b_n\}$ defined by

$a_n = \frac{1}{n} \in \mathbb{Q}$, $b_n = \frac{\sqrt{2}}{n} \in \mathbb{R}\setminus\mathbb{Q}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$

$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} 0 = 0$

$\therefore \lim_{x\to 0} f(x)$ does NOT exist.

Actually, with little modification, we can show $\lim_{x\to c} f(x)$ does NOT exist $\forall c \in \mathbb{R}$.

Exercise 3.3.1

Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $\lim_{x\to 0} f(x)$ does NOT exist.

Hint: Consider sequences $\{a_n\}$, $\{b_n\}$ defined by

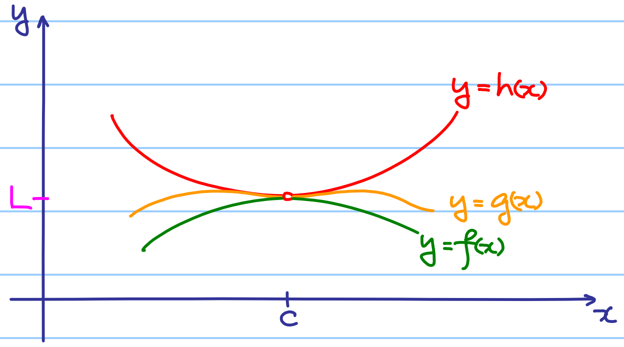
$$a_n = \frac{1}{2n\pi} \quad , \quad b_n = \frac{1}{(2n+\frac{1}{2})\pi}$$

3.4 Sandwich Theorem for Functions

Theorem 3.4.1

If $f(x) \le g(x) \le h(x)$ $\forall x \in \mathbb{R} \setminus \{c\}$ and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Geometrical meaning:



In fact, the result is still true if $f(x) \le g(x) \le h(x)$ holds in an open interval containing c but possibly except c.

Example 3.4.1

Prove that $\lim_{x\to 0} x^2\cos^2\frac{1}{x} = 0$

Note that $0 \le x^2\cos^2\frac{1}{x} \le x^2$ and $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$

By sandwich theorem, $\lim_{x\to 0} x^2\cos^2\frac{1}{x} = 0$.

Remark:

Sandwich theorem can be generalized to left and right hand limit.

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions and $c \in \mathbb{R}$

If $f(x) \le g(x) \le h(x)$ for all $x < c$ $(x > c)$ and $\lim_{x\to c^-} f(x) = \lim_{x\to c^-} h(x) = L$ $\left(\lim_{x\to c^+} f(x) = \lim_{x\to c^+} h(x) = L\right)$

then $\lim_{x\to c^-} g(x) = L$ $\left(\lim_{x\to c^+} g(x) = L\right)$.

Theorem 3.4.2

$\lim_{x \to c} f(x) = 0 \Leftrightarrow \lim_{x \to c} |f(x)| = 0$

proof :

"$\Leftarrow$" Suppose that $\lim_{x \to c} |f(x)| = 0$

Note that $-|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in \mathbb{R} \setminus \{c\}$ and $\lim_{x \to c} -|f(x)| = \lim_{x \to c} |f(x)| = 0$

by the sandwich theorem, $\lim_{x \to c} f(x) = 0$

"$\Rightarrow$" Suppose that $\lim_{x \to c} f(x) = 0$.

Then $\lim_{x \to c} [f(x)]^2 = (\lim_{x \to c} f(x)) \cdot (\lim_{x \to c} f(x)) = 0 \cdot 0 = 0$

Note that $|f(x)| = \sqrt{[f(x)]^2}$

$$\begin{aligned} \therefore \lim_{x \to c} |f(x)| &= \lim_{x \to c} \sqrt{[f(x)]^2} \\ &\overset{(*)}{=} \sqrt{\lim_{x \to c} [f(x)]^2} \\ &= \sqrt{0} \\ &= 0 \end{aligned}$$

(*) is true because of $\sqrt{x}$ is a function that is continuous at 0.

Example 3.4.2

By considering the previous theorem with $f(x) = x$, we have $\lim_{x \to 0} x = 0 \Leftrightarrow \lim_{x \to 0} |x| = 0$.

Example 3.4.3

Prove that $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

Note that $-1 \leq \cos \frac{1}{x} \leq 1 \quad \forall x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$

$-|x| \leq x \leq |x| \quad \forall x \in \mathbb{R}$

$\therefore \quad -|x| \leq x \cos \frac{1}{x} \leq |x| \quad \forall x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$

Also $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$,

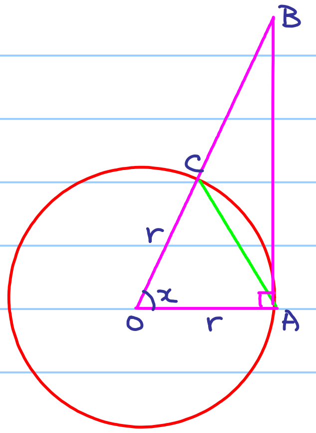
by the sandwich theorem, $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

Theorem 3.4.3

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

proof:

1) Consider $0 < x < \frac{\pi}{2}$, we have



Area of $\triangle OAC$ < Area of sector OAC < Area of $\triangle OAB$

$$\frac{1}{2}r^2\sin x < \frac{1}{2}r^2 x < \frac{1}{2}r^2\tan x$$

$$\sin x < x < \tan x$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\sin x}{x} < 1 \qquad\qquad \cos x < \frac{\sin x}{x}$$

$$\therefore \cos x < \frac{\sin x}{x} < 1$$

2) Consider $-\frac{\pi}{2} < x < 0$, we have

Let $y = -x$, then $0 < y < \frac{\pi}{2}$, so

$$\cos y < \frac{\sin y}{y} < 1$$

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1$$

$$\therefore \cos x < \frac{\sin x}{x} < 1$$

$\therefore$ By (1) and (2) , $\cos x < \frac{\sin x}{x} < 1 \quad \forall x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$

Also $\lim_{x\to 0}\cos x = \lim_{x\to 0} 1 = 1$,

by the sandwich theorem, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Example 3.4.4

Find $\lim_{x\to 0}\frac{\sin 3x}{2x}$.

$$\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3}{2} = 1\cdot\frac{3}{2} = \frac{3}{2}$$

Example 3.4.5

Find $\lim_{x\to 0}\frac{\cos ax - \cos bx}{x^2}$

$$\lim_{x\to 0}\frac{\cos ax - \cos bx}{x^2} = \lim_{x\to 0}\frac{2\sin\frac{a+b}{2}x\,\sin\frac{b-a}{2}x}{x^2}$$

$$= \lim_{x\to 0} 2\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)\frac{\sin\frac{a+b}{2}x}{\frac{a+b}{2}x}\;\frac{\sin\frac{b-a}{2}x}{\frac{b-a}{2}x}$$

$$= \frac{b^2-a^2}{2}$$

3.5 Limits at Infinity

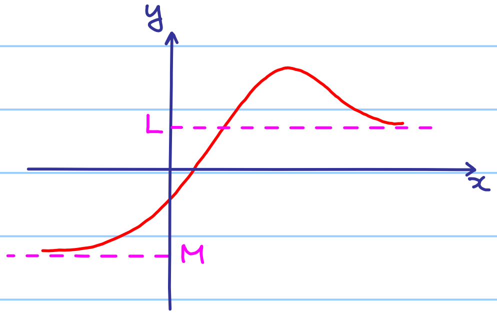
Definition 3.5.1 (Informal)

If f(x) gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of f(x) at $+\infty$. We write $\lim_{x\to+\infty} f(x) = L$.

(Similar definition for $\lim_{x\to-\infty} f(x)$)

From the graph, we have

$\lim_{x\to+\infty} f(x) = L$ but $\lim_{x\to-\infty} f(x) = M$.



$\therefore$ $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are NOT necessary to be the same!

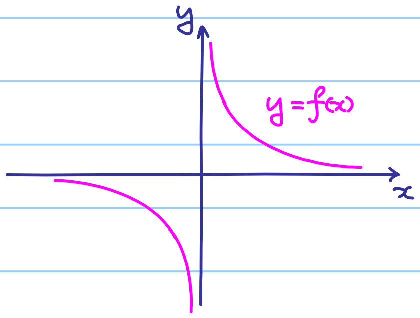
However if $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = L$, some simply write $\lim_{x\to\pm\infty} f(x) = L$.

Example 3.5.1

Let $f(x) = \frac{1}{x}$, $x \neq 0$.

Then $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$,

or simply write $\lim_{x\to\pm\infty} f(x) = 0$.



Theorem 3.5.1

1) If $k > 0$, then $\lim_{x\to+\infty} \frac{1}{x^k} = 0$.

2) $\lim_{x\to+\infty} (1+\frac{1}{x})^x = e$

(NOT surprising as $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$)

3.6 Algebraic Properties of Limits at Infinity

Theorem 3.6.1

If both $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist (Very important assumption!), then

(1) $\lim_{x\to+\infty} f(x)+g(x) = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x)$

(2) $\lim_{x\to+\infty} f(x)-g(x) = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x)$

(3) $\lim_{x\to+\infty} f(x)g(x) = \lim_{x\to+\infty} f(x) \cdot \lim_{x\to+\infty} g(x)$

(4) $\lim_{x\to+\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ if $\lim_{x\to+\infty} g(x) \neq 0$.

Similar results hold for limits at $-\infty$.

Example 3.6.1

Find $\lim_{x\to+\infty} \dfrac{3x^2}{x^2+x+1}$

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1} \neq \frac{\lim_{x\to+\infty} 3x^2}{\lim_{x\to+\infty} x^2+x+1}$$

Both limits does NOT exist.

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \lim_{x\to+\infty} \frac{3}{1+0+0}$$

$$= 3$$

Example 3.6.2

Find $\lim_{x\to+\infty} \dfrac{2x+1}{3x^2-2x+1}$

$$\lim_{x\to+\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3-\frac{2}{x}+\frac{1}{x^2}}$$

$$= \frac{0+0}{3+0+0}$$

$$= 0$$

Conclusion :

If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$ (i.e. $\deg p(x) = m$)

$q(x) = b_n x^n + a_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ with $b_n \neq 0$ (i.e. $\deg q(x) = n$)

then

$$\lim_{x \to +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty / -\infty & \text{if } m > k \\ \dfrac{a_m}{b_m} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Similar result as the case in limits of sequences !

Example 3.6.3

Find $\displaystyle\lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+1}}$

$$\lim_{x \to -\infty} \frac{x}{\sqrt{4x^2+1}}$$

deg 1

roughly, deg 1

$\Rightarrow$ limit should exist !

$$= \lim_{x \to -\infty} \frac{1}{\frac{1}{x}\sqrt{4x^2+1}}$$

$$= \lim_{x \to -\infty} \frac{1}{-\sqrt{\frac{1}{x^2}} \cdot \sqrt{4x^2+1}}$$

(Caution : $x < 0 \Rightarrow \frac{1}{x} = -\sqrt{\left(\frac{1}{x}\right)^2} = -\sqrt{\frac{1}{x^2}}$)

$$= \lim_{x \to -\infty} -\frac{1}{\sqrt{4 + \frac{1}{x^2}}}$$

$$= -\frac{1}{2}$$

Following this idea, we are going to compare exponential functions and polynomials.

Theorem 3.6.2

1) $\displaystyle\lim_{x \to +\infty} x^k e^{-x} = \lim_{x \to +\infty} \frac{x^k}{e^x} = 0$, for any $k > 0$.

2) $\displaystyle\lim_{x \to +\infty} p(x) e^{-x} = \lim_{x \to +\infty} \frac{p(x)}{e^x} = 0$, for any polynomial $p(x)$.

Roughly speaking : As $x \to +\infty$, e^x grows "faster" than any polynomial

Proof can be done when L'Hôptial's rule is covered.

3.7 Limits Involving e

Example 3.7.1

Find $\lim_{x\to+\infty}(1+\frac{1}{2x-1})^{x}$

$$\lim_{x\to+\infty}(1+\frac{1}{2x-1})^{x} = \lim_{x\to+\infty}(1+\frac{1}{2x-1})^{\frac{1}{2}(2x-1)+\frac{1}{2}}$$

$$= \lim_{x\to+\infty}[(1+\frac{1}{2x-1})^{2x-1}]^{\frac{1}{2}} \cdot (1+\frac{1}{2x-1})^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} \cdot 1$$

$$= e^{\frac{1}{2}}$$

Example 3.7.2

Find $\lim_{x\to-\infty}(1+\frac{1}{x})^{x}$

Let $y=-x$, as $x\to-\infty$, $y\to+\infty$

$$\lim_{x\to-\infty}(1+\frac{1}{x})^{x} = \lim_{y\to+\infty}(1-\frac{1}{y})^{-y}$$

$$= \lim_{y\to+\infty}(\frac{y}{y-1})^{y}$$

$$= \lim_{y\to+\infty}(1+\frac{1}{y-1})^{y-1}\cdot(1+\frac{1}{y-1})$$

$$= e\cdot 1$$

$$= e$$

Remark: From the above example, we know $\lim_{x\to\pm\infty}(1+\frac{1}{x})^{x}=e$.

Example 3.7.3

Find $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$.

Let $y=\frac{1}{x}$, as $x\to 0$, $y\to\pm\infty$ (Not only $+\infty$, but also $-\infty$)

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{y\to\pm\infty}(1+\frac{1}{y})^{y} = e$$

3.8 Sandwich Theorem at Infinity

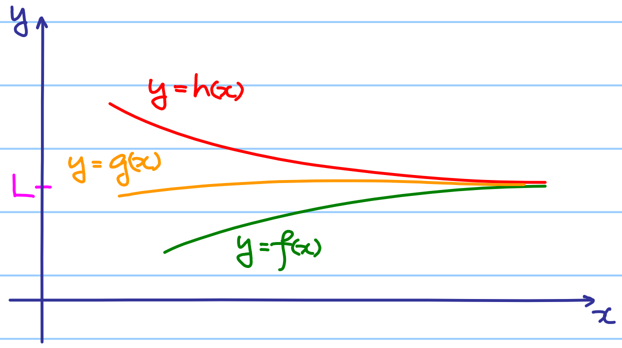
Theorem 3.8.1

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions.

If $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$ (actually: $[a, +\infty)$ is sufficient)

and $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} h(x) = L$, then $\lim_{x \to +\infty} g(x) = L$.

Geometrical meaning:



Similar result holds for limits at $-\infty$.

Example 3.8.1

Find $\lim_{x \to +\infty} e^{-x} \sin x$

Since $-1 \leq \sin x \leq 1$ and $e^{-x} > 0$

$$-e^{-x} \leq e^{-x} \sin x \leq e^{-x}$$

Note: $\lim_{x \to +\infty} -e^{-x} = \lim_{x \to +\infty} e^{-x} = 0$.

By the sandwich theorem, $\lim_{x \to +\infty} e^{-x} \sin x = 0$.